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The Elliptic Representation of the Painlevé 6 Equation

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1 Introduction

We review our results, to be found in [10] [11], on the elliptic representation of the sixth Painlevé equation

$$\frac{d^2y}{dx^2}=\frac{1}{2}\left[\frac{1}{y}+\frac{1}{y-1}+\frac{1}{y-x}\right]\left(\frac{dy}{dx}\right)^2-\left[\frac{1}{x}+\frac{1}{x-1}+\frac{1}{y-x}\right]\frac{dy}{dx}$$

$$+\frac{y(y-1)(y-x)}{x^2(x-1)^2}\left[\alpha+\beta\frac{x}{y^2}+\gamma\frac{x-1}{(y-1)^2}+\delta\frac{x(x-1)}{(y-x)^2}\right], \qquad \text{(PVI)}.$$

Though the elliptic representation of PVI has been known since R.Fuchs [7], in the literature there is no general study of its analytic implications. To fill this gap, we studied in [11] the analytic properties of the solutions in elliptic representation for all values of $\alpha,\beta,\gamma,\delta$ and we derived their critical behavior close to the singular points $x=0,1,\infty$. Moreover, we solved the connection problem for generic values of $\alpha,\beta,\gamma,\delta$ and in [10] for the special (non-generic) case $\beta=\gamma=1-2\delta=0$, which is important in 2-D topological field theory.

The first analytical problem with Painlevé equations is to determine the critical behavior of the transcendents at the critical points $x=0,1,\infty$. Such a behavior must depend on two parameters, which are integration constants. The second problem, called *connection problem*, is to find the relation between the couples of parameters at different critical points. The method of isomonodromic deformations developed in [14] [15] was applied to the Painlevé 6 equation in [13], to solve such problems for a class of solutions of PVI with generic values of the parameters. The non-generic case $\beta=\gamma=1-2\delta=0$ is studied in [6] [19] [10] for its applications to topological field theory. Studies on the critical behavior can be also found in [25].

Here we show that the elliptic representation is a valuable tool to study the critical behavior of the Painlevé 6 transcendents. In [10] [11] we obtained results which include the results of [13] [6] and extend the class of solutions to which they apply. On the other hand, we needed to use the isomonodromic deformation theory to solve the connection problem, to be formulated below, for the elliptic representation.

The elliptic representation was introduced by P. Painlevé in [22] and R. Fuchs in [7]. Let

$$\mathcal{L}:=x(1-x)\frac{d^2}{dx^2}+(1-2x)\frac{d}{dx}-\frac{1}{4}.$$

be a linear differential operator and let $\wp(z;\omega_1,\omega_2)$ be the Weierstrass elliptic function of the independent variable $z\in\mathbf{P}^1$, with *half-periods* ω_1, ω_2. Let us consider the following independent solutions of the *hyper-geometric equation* $\mathcal{L}\omega=0$:

$$\omega_1(x):=\frac{\pi}{2}F\left(\frac{1}{2},\frac{1}{2},1;x\right),\qquad \omega_2(x):=i\frac{\pi}{2}F\left(\frac{1}{2},\frac{1}{2},1;1-x\right),$$

where $F\left(\frac{1}{2},\frac{1}{2},1;x\right)$ is the standard notation for the hyper-geometric function. Here x is in the universal covering of $\mathbf{P}^1\backslash\{0,1,\infty\}$, so that at this stage we do not worry about the choice of branch-cuts. It is proved in [7] that the Painlevé 6 equation is equivalent to the following differential equation for a new function $u(x)$:

$$\mathcal{L}(u)=\frac{1}{2x(1-x)}\frac{\partial}{\partial u}\left\{2\alpha\left[\wp\left(\frac{u}{2};\omega_1,\omega_2\right)+\frac{1+x}{3}\right]-2\beta\frac{x}{\wp\left(\frac{u}{2};\omega_1,\omega_2\right)+\frac{1+x}{3}}+\right.$$

$$\left.+2\gamma\frac{1-x}{\wp\left(\frac{u}{2};\omega_1,\omega_2\right)+\frac{x-2}{3}}+(1-2\delta)\frac{x(1-x)}{\wp\left(\frac{u}{2};\omega_1,\omega_2\right)+\frac{1-2x}{3}}\right\} \qquad (1)$$

The connection to Painlevé 6 is given by the following representation of the transcendents:

$$y(x) = \wp\left(\frac{u(x)}{2}; \omega_1(x), \omega_2(x)\right) + \frac{1+x}{3}.$$

The algebraic-geometrical properties of the elliptic representations where studied in [18]. Nevertheless, the analytic properties of the function $u(x)$ were not studied, except for the special case $\alpha = \beta = \gamma = 1 - 2\delta = 0$. In this case the function $u(x)$ is a linear combination of ω_1 and ω_2. This case was well known to Picard [23], and the critical behavior was studied in [19].

In [11], we studied the analytic properties of $u(x)$ for *any* value of $\alpha, \beta, \gamma, \delta$. As a result, given a Painlevé 6 equation specified by a choice of $\alpha, \beta, \gamma, \delta$, we found the critical behavior of its transcendents belonging to a class which contains almost all possible solutions of the equation. The meaning of "almost" will be clear later. A transcendent in the class vanishes as x (as as variable in the universal covering of $\mathbf{P}^1 \backslash \{0, 1, \infty\}$) approaches a critical point. Nevertheless, along some particular paths approaching the critical point, the transcendent does not vanish: it has oscillatory behavior. Qualitatively speaking, the oscillations are due to the existence of (movable) poles close to the particular paths having an accumulation point in the critical point. In [10] we found analogous results for the special case $\beta = \gamma = 1 - 2\delta = 0$ and α any complex number

As remarked above, our class of solutions include "almost" all transcendents, but there are some transcendents which are not singed out by our method. This is for example the case of the *Chazy solutions*, whose critical behavior is different from ours (see [19]).

2 Our results

2.1 Local Representation

The equation $\mathcal{L}(u) = 0$ has a general solution $u_0(x) = 2\nu_1\omega_1(x) + 2\nu_2\omega_2(x)$, $\nu_1, \nu_2 \in \mathbf{C}$. We look for a solution of (1) of the form $u(x) = 2\nu_1\omega_1(x) + 2\nu_2\omega_2(x) + 2v(x)$, where $v(x)$ is a perturbation of u_0. Let $\mathbf{C}_0 := \mathbf{C}\backslash\{0\}$, $\widetilde{\mathbf{C}_0}$ the universal covering and let $0 < r < 1$. We define the domains

$$\mathcal{D}(r; \nu_1, \nu_2) := \left\{ x \in \widetilde{\mathbf{C}_0} \text{ such that } |x| < r, \left| \frac{e^{-i\pi\nu_1}}{16^{1-\nu_2}} x^{1-\nu_2} \right| < r, \left| \frac{e^{i\pi\nu_1}}{16^{\nu_2}} x^{\nu_2} \right| < r \right\} \tag{2}$$

$$\mathcal{D}_0(r) := \left\{ x \in \widetilde{\mathbf{C}_0} \text{ such that } |x| < r \right\} \tag{3}$$

We observe that the translations $\nu_i \mapsto \nu_i + 2N_i$, $i = 1, 2$, $N_i \in \mathbf{Z}$ do not change a transcendent in the elliptic representation

$$y(x) = \wp\left(\nu_1\omega_1(x) + \nu_2\omega_2(x) + v(x); \omega_1(x), \omega_2(x)\right) + \frac{1+x}{3}.$$

This is a consequence of the periodicity of the $\wp$-function. Therefore, one can take $0 \leq \Re\nu_i < 2$, $i = 1, 2$. Nevertheless, we don't need to suppose such a range explicitly. Only in the case $\Im\nu_2 = 0$ we need to suppose that $0 \leq \nu_2 < 2$. Finally, let us introduce the following expansion:

$$v(x; \nu_1, \nu_2) := \sum_{n \geq 1} a_n x^n + \sum_{n \geq 0, m \geq 1} b_{nm} x^n \left[e^{-i\pi\nu_1} \left(\frac{x}{16}\right)^{1-\nu_2} \right]^m + \sum_{n \geq 0, m \geq 1} c_{nm} x^n \left[e^{i\pi\nu_1} \left(\frac{x}{16}\right)^{\nu_2} \right]^m \tag{4}$$

Theorem 1: *Let ν_1, ν_2 be two complex numbers.*

I) *For any complex ν_1, ν_2 such that $\Im\nu_2 \neq 0$ there exist a positive number $r < 1$ and a transcendent*

$$y(x) = \wp\Big(\nu_1\omega_1(x) + \nu_2\omega_2(x) + v(x; \nu_1, \nu_2);\ \omega_1(x), \omega_2(x)\Big) + \frac{1+x}{3}$$

such that $v(x; \nu_1, \nu_2)$ is holomorphic in the domain $\mathcal{D}(r; \nu_1, \nu_2)$ and it is given by the expansion (4) which is convergent in $\mathcal{D}(r; \nu_1, \nu_2)$. The coefficients a_n, b_{nm}, c_{nm}, $i = 1, 2$, are certain rational functions of ν_2. Moreover, there exists a positive constant $M(\nu_2)$ such that

$$|v(x; \nu_1, \nu_2)| \leq M(\nu_2) \left(|x| + \left| e^{-i\pi\nu_1} \left(\frac{x}{16}\right)^{1-\nu_2} \right| + \left| e^{i\pi\nu_1} \left(\frac{x}{16}\right)^{\nu_2} \right| \right) \quad \text{in } \mathcal{D}(r; \nu_1, \nu_2) \tag{5}$$

II) *For any complex ν_1 and real ν_2, with the constraint $0 < \nu_2 < 1$ or $1 < \nu_2 < 2$, there exists a positive $r < 1$ and a transcendent*

$$y(x) = \wp\Big(\nu_1\omega_1(x) + \nu_2\omega_2(x) + v(x;\nu_1,\nu_2);\ \omega_1(x),\omega_2(x)\Big) + \frac{1+x}{3}, \quad \textit{if } 0 < \nu_2 < 1$$

or

$$y(x) = \wp\Big(\nu_1\omega_1(x) + \nu_2\omega_2(x) + v(x;-\nu_1,2-\nu_2);\ \omega_1(x),\omega_2(x)\Big) + \frac{1+x}{3}, \quad \textit{if } 1 < \nu_2 < 2$$

such that $v(x;\nu_1,\nu_2)$ and $v(x;-\nu_1,2-\nu_2)$ are holomorphic in $\mathcal{D}_0(r)$, with convergent expansion (4) and bound (5) (for $1 < \nu_2 < 2$ substitute $\nu_1 \mapsto -\nu_1$, $\nu_2 \mapsto 2-\nu_2$).

Note that in the theorem

$$\nu_2 \neq 0, 1$$

We stress that in case **II)**, if ν_2 is greater that 2 or less then 0, we can always make a translation $\nu_2 \mapsto \nu_2 + 2N$ to obtain $0 < \nu_2 < 2$ (on the other hand, if $-2N < \nu_2 < 2-2N$, the formulae of case **II)** hold with the substitution $\nu_2 \mapsto \nu_2 + 2N$). Note also that ν_1 and ν_2 play asymmetric roles.

Observation 1: As a consequence of the theorem, for any $N \in \mathbf{Z}$ and for any complex ν_1, ν_2 such that $\Im\nu_2 \neq 0$, there exists $r_N < 1$ and a transcendent $y(x) = \wp\Big(\nu_1\omega_1(x) + [\nu_2 + 2N]\omega_2(x) + v(x;\nu_1,\nu_2 + 2N);\ \omega_1(x),\omega_2(x)\Big) + \frac{1+x}{3}$ in $\mathcal{D}(r;\nu_1,\nu_2+2N)$. By periodicity of the $\wp$-function we re-write the transcendent as follows:

$$y(x) = \wp\Big(\nu_1\omega_1(x) + \nu_2\omega_2(x) + v(x;\nu_1,\nu_2+2N);\ \omega_1(x),\omega_2(x)\Big) + \frac{1+x}{3} \quad \text{in } \mathcal{D}(r;\nu_1,\nu_2+2N).$$

Moreover, we showed in [11] that if a transcendent has the elliptic representation

$$y(x) = \wp\Big(\nu_1\omega_1(x) + \nu_2\omega_2(x) + v(x;\nu_1,\nu_2);\ \omega_1(x),\omega_2(x)\Big) + \frac{1+x}{3}$$

in $\mathcal{D}(r,\nu_1,\nu_2)$ for some ν_1,ν_2, $\Im\nu_2 \neq 0$, then for any integer N there exists ν_1' (depending on ν_1, ν_2 and N) such that the transcendent has also the representation

$$y(x) = \wp\Big(\nu_1'\omega_1(x) + \nu_2\omega_2(x) + v(x;\nu_1,\nu_2+2N);\ \omega_1(x),\omega_2(x)\Big) + \frac{1+x}{3}$$

in $\mathcal{D}(r,\nu_1',\nu_2+2N)$. ν_1' can be explicitly computed.

Observation 2: Another consequence of the theorem is that for any complex ν_1,ν_2 such that $\Im\nu_2 \neq 0$ there exists $y(x) = \wp\Big(-\nu_1\omega_1(x) + [2-\nu_2]\,\omega_2(x) + v(x;-\nu_1,2-\nu_2);\ \omega_1(x),\omega_2(x)\Big) + \frac{1+x}{3}$. Again we use the fact that the $\wp$-function is periodic w.r.t. $2\omega_2$ and it is an even function. Therefore the transcendent becomes

$$y(x) = \wp\Big(\nu_1\omega_1(x) + \nu_2\omega_2(x) - v(x;-\nu_1,2-\nu_2);\ \omega_1(x),\omega_2(x)\Big) + \frac{1+x}{3}, \quad \text{in } \mathcal{D}(r;-\nu_1,2-\nu_2)$$

Note that the series $-v(x;-\nu_1,2-\nu_2)$ is of the form

$$\sum_{n\geq 1} a_n x^n + \sum_{n\geq 0, m\geq 1} b_{nm}x^n \left[e^{-i\pi\nu_1}\left(\frac{x}{16}\right)^{2-\nu_2}\right]^m + \sum_{n\geq 0,m\geq 1} c_{nm}x^n \left[e^{i\pi\nu_1}\left(\frac{x}{16}\right)^{\nu_2-1}\right]^m$$

where we have re-named the constants a_n, b_{nm}, c_{nm}.

The domain $\mathcal{D}(r_N;\nu_1,\nu_2+2N)$ can be written as follows:

$$(\Re\nu_2 + 2N)\ln\frac{|x|}{16} - \pi\Im\nu_1 - \ln r_N < \Im\nu_2 \arg x <$$

$$< (\Re\nu_2 - 1 + 2N)\ln\frac{|x|}{16} - \pi\Im\nu_1 + \ln r_N, \quad |x| < r_N$$

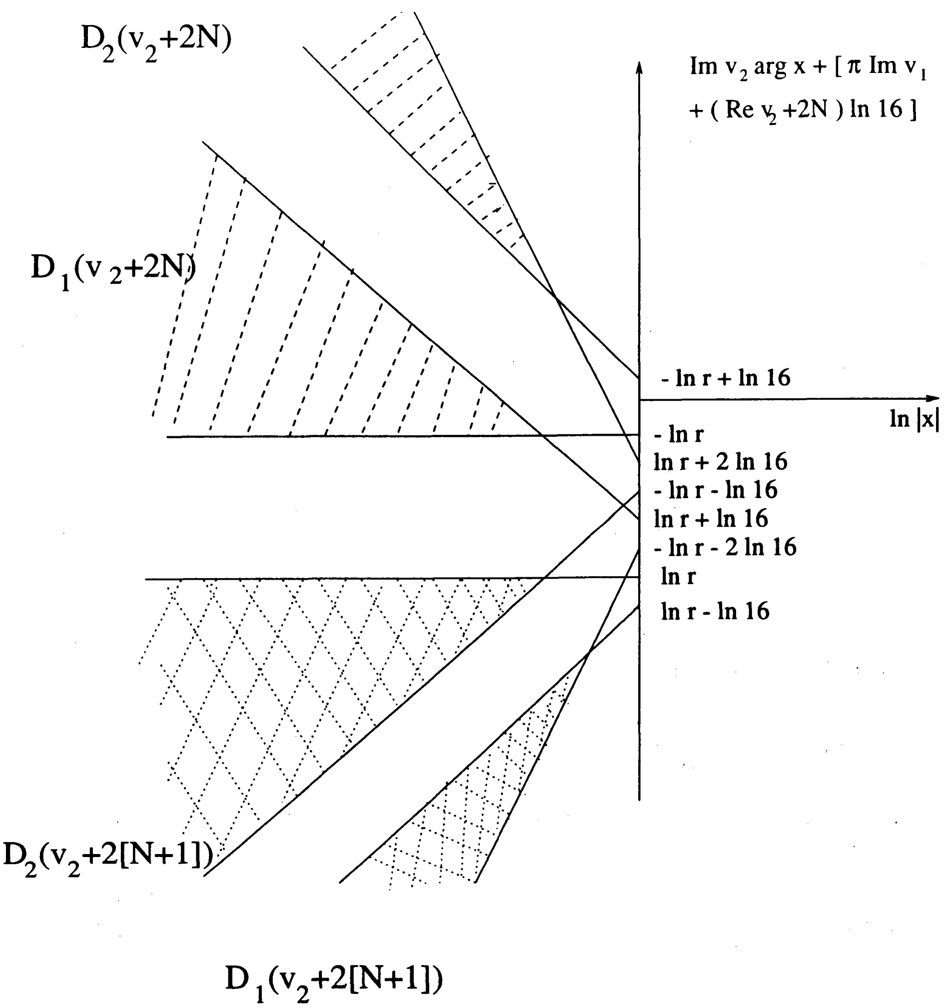

Figure 1: The domains $\mathcal{D}_1(r;\nu_1,\nu_2+2N) := \mathcal{D}(r;\nu_1,\nu_2+2N)$, $\mathcal{D}_2(r;\nu_1,\nu_2+2N) := \mathcal{D}(r;-\nu_1,2-\nu_2-2N)$ and $\mathcal{D}_1(r;\nu_1,\nu_2+2[N+1])$, $\mathcal{D}_2(r;\nu_1,\nu_2+2[N+1])$ for arbitrarily fixed values of ν_1, ν_2, N. They are represented in the plane $(\ln|x|,\ \Im\nu_2 \arg x + [\pi\Im\nu_1 + (\Re\nu_2+2N)\ln 16])$.

Therefore the domain $\mathcal{D}(r_N,-\nu_1,2-\nu_2-2N)$ is

$$(\Re\nu_2 - 1 + 2N)\ln\frac{|x|}{16} - \pi\Im\nu_1 - \ln r_N < \Im\nu_2 \arg x <$$

$$< (\Re\nu_2 - 2 + 2N)\ln\frac{|x|}{16} - \pi\Im\nu_1 + \ln r_N, \quad |x| < r_N$$

We can draw their picture in the $(\ln|x|, \Im\nu_2 \arg x)$-plane. See figure 1.

It is remarkable that the elliptic representation allows us to conclude that the same transcendent has different representations on the union of the domains $\mathcal{D}(r_N,-\nu_1,2-\nu_2-2N)$, $\mathcal{D}(r_N;\nu_1,\nu_2+2N)$. The movable poles of the transcendent are outside the union.

2.2 Critical Behavior

It is possible to compute the critical behavior for $x \to 0$ of a transcendent of Theorem 1. For simplicity, we consider $x \to 0$ along the paths defined below. Let $\Im\nu_2 \neq 0$ and $\mathcal{V} \in \mathbf{C}$. We define the following family of paths joining a point $x_0 \in \mathcal{D}(r;\nu_1,\nu_2)$ to $x=0$

$$\arg x = \arg x_0 + \frac{\Re\nu_2 - \mathcal{V}}{\Im\nu_2}\ln\frac{|x|}{|x_0|}, \quad 0 \le \mathcal{V} \le 1 \tag{6}$$

The paths are contained in $\mathcal{D}(r;\nu_1,\nu_2)$. If $\Im\nu_2 = 0$ any regular path contained in $\mathcal{D}_0(r)$ can be considered.

Theorem 2: *Let ν_1, ν_2 be given.*

If $\Im\nu_2 \neq 0$, the critical behavior of the transcendent $y(x) = \wp(\nu_1\omega_1 + \nu_2\omega_2 + v(x;\nu_1,\nu_2);\omega_1,\omega_2) + (1+x)/3$ when $x \to 0$ along the path (6) is:

For $0 < \mathcal{V} < 1$:

$$y(x) = -\frac{1}{4}\left[\frac{e^{i\pi\nu_1}}{16^{\nu_2-1}}\right] x^{\nu_2} \left(1 + O(|x^{\nu_2}| + |x^{1-\nu_2}|)\right). \tag{7}$$

For $\mathcal{V} = 0$:

$$y(x) = \left[\frac{x}{2} + \sin^{-2}\left(-i\frac{\nu_2}{2}\ln\frac{x}{16} + \frac{\pi\nu_1}{2} + \sum_{m\geq 1} c_{0m}\left[e^{i\pi\nu_1}\left(\frac{x}{16}\right)^{\nu_2}\right]^m\right)\right] (1 + O(x)). \tag{8}$$

For $\mathcal{V} = 1$:

$$y(x) = x\,\sin^2\left(i\frac{1-\nu_2}{2}\ln\frac{x}{16} + \frac{\pi\nu_1}{2} + \sum_{m\geq 1} b_{0m}\left[e^{-i\pi\nu_1}\left(\frac{x}{16}\right)^{1-\nu_2}\right]^m\right) (1 + O(x)). \tag{9}$$

For ν_2 real we have two cases. For $0 < \nu_2 < 1$, the transcendent $y(x) = \wp(\nu_1\omega_1+\nu_2\omega_2+v(x;\nu_1,\nu_2);\omega_1,\omega_2)+(1+x)/3$ defined in $\mathcal{D}_0(r)$ has behavior

$$y(x) = -\frac{1}{4}\left[\frac{e^{i\pi\nu_1}}{16^{\nu_2-1}}\right] x^{\nu_2} \left(1 + O(|x^{\nu_2}| + |x^{1-\nu_2}|)\right), \quad 0 < \nu_2 < 1 \tag{10}$$

For $1 < \nu_2 < 2$, the transcendent $y(x) = \wp(\nu_1\omega_1 + \nu_2\omega_2 + v(x;-\nu_1, 2-\nu_2);\omega_1,\omega_2) + (1+x)/3$ defined in $\mathcal{D}_0(r)$ has behavior

$$y(x) = -\frac{1}{4}\left[\frac{e^{i\pi\nu_1}}{16^{\nu_2-1}}\right]^{-1} x^{2-\nu_2} \left(1 + O(|x^{2-\nu_2}| + |x^{\nu_2-1}|)\right), \quad 1 < \nu_2 < 2 \tag{11}$$

Note that for $\mathcal{V} = 0$ the transcendent has oscillatory behavior with no limit as $x \to 0$. The oscillations are due the existence of poles that lie outside the union of the domains of figure 1. They have an accumulation point in the critical point $x = 0$. In [11] we showed the existence of such poles in one example for $\alpha = \beta = \gamma = 1 - 2\delta = 0$.

2.3 The Critical Points $x = 1, \infty$

Theorems 1 and 2 deal with the point $x = 0$. We now turn to the other critical points. Let us use the notation $\omega_1^{(0)} := \omega_1$, $\omega_2^{(0)} := \omega_2$; they are a basis of solutions for the hyper-geometric equation at $x = 0$. Let us define $\omega_1^{(1)} := \omega_2$, $\omega_2^{(1)} := \omega_1$: they are a basis of solutions for the hyper-geometric equation at $x = 1$. Finally, let $\omega_1^{(\infty)} := \omega_1 + \omega_2$, $\omega_2^{(\infty)} := \omega_2$: they are a basis of solutions for the hyper-geometric equation at $x = \infty$. We construct solutions

$$\frac{u(x)}{2} = \nu_1^{(1)}\omega_1^{(1)}(x) + \nu_2^{(1)}\omega_2^{(1)}(x) + v^{(1)}(x)$$

in a neighborhood of $x = 1$, and solutions

$$\frac{u(x)}{2} = \nu_1^{(\infty)}\omega_1^{(\infty)}(x) + \nu_2^{(\infty)}\omega_2^{(\infty)}(x) + v^{(\infty)}(x)$$

in a neighborhood of $x = \infty$. For the computation of the critical behaviors of $u(x)$ we need the connection formulas for the three bases of solutions of the hyper-geometric equation (see [20]). Thus, it is necessary to specify branch-cuts in the above definitions. We choose $|\arg x| < \pi$ for $\omega_1^{(1)}$, $|\arg(1-x)| < \pi$ for $\omega_2^{(1)}$, $-\pi < \arg x < 0$ for $\omega_1^{(\infty)}$ and $|\arg x| < \pi$ for $\omega_2^{(\infty)}$. Once they are so defined, they are continued on the universal covering of $\mathbf{P}^1\backslash\{0,1,\infty\}$.

We refer to [11] for the analogous of Theorems 1 and 2 at $x = 1, \infty$.

2.4 Connection Problem

The elliptic representation allows us to obtained detailed information about the critical behavior of the Painlevé transcendents. On the other hand, the local analysis does not solve the *connection problem.* This is the problem of determining the critical behavior of a given transcendent at $x = 0$, $x = 1$ and $x = \infty$. In our framework, we ask if a transcendent may have, at the same time, three representations

$$\begin{aligned} y(x) &= \wp(\nu_1^{(0)}\omega_1^{(0)} + \nu_2^{(0)}\omega_2^{(0)} + v^{(0)}) + \frac{1+x}{3} \\ &= \wp(\nu_1^{(1)}\omega_1^{(1)} + \nu_2^{(1)}\omega_2^{(1)} + v^{(1)}) + \frac{1+x}{3} \\ &= \wp(\nu_1^{(\infty)}\omega_1^{(\infty)} + \nu_2^{(\infty)}\omega_2^{(\infty)} + v^{(\infty)}) + \frac{1+x}{3}. \end{aligned}$$

Moreover, we look for formulae which connect the three couples of parameters $(\nu_1^{(0)}, \nu_2^{(0)})$, $(\nu_1^{(1)}, \nu_2^{(1)})$, $(\nu_1^{(\infty)}, \nu_2^{(\infty)})$.

The connection problem may be solved using the method of isomonodromic deformations, as it was first done in [13]. The PVI is the isomonodromy deformation equation of a Fuchsian system of differential equations

$$\frac{dY}{dz} = \left[\frac{A_0(x)}{z} + \frac{A_x(x)}{z-x} + \frac{A_1(x)}{z-1}\right] Y$$

The 2×2 matrices $A_i(x)$ ($i = 0, x, 1$ are labels) depend on x in such a way that the monodromy of a fundamental solution $Y(z, x)$ does not change for small deformations of x. They depend on the parameters $\alpha, \beta, \gamma, \delta$ of PVI as follows:

$$A_0(x) + A_1(x) + A_x(x) = -\frac{1}{2}\begin{pmatrix} \theta_\infty & 0 \\ 0 & -\theta_\infty \end{pmatrix}, \qquad \text{eigenvalues of } A_i(x) = \pm\frac{1}{2}\theta_i, \qquad i = 0, 1, x$$

$$\alpha = \frac{1}{2}(\theta_\infty - 1)^2, \quad \beta = -\frac{1}{2}\theta_0^2, \quad \gamma = \frac{1}{2}\theta_1^2, \quad \delta = \frac{1}{2}(1 - \theta_x^2)$$

In [11] we solved the connection problem for the elliptic representation for generic values of α, β, γ, δ . More precisely, by *generic case* we mean:

$$\nu_2^{(i)},\ \theta_0,\ \theta_x,\ \theta_1,\ \theta_\infty \not\in \mathbf{Z}; \qquad \frac{\pm 1 \pm \nu_2^{(i)} \pm \theta_1 \pm \theta_\infty}{2},\ \frac{\pm 1 \pm \nu_2^{(i)} \pm \theta_0 \pm \theta_x}{2} \not\in \mathbf{Z} \tag{12}$$

The signs $\pm$ vary independently. This is a technical condition which can be abandoned (except for $\nu_2^{(i)} \not\in \mathbf{Z}$) at the price of making the computations more complicated. For example, the non-generic case $\beta = \gamma = 1 - 2\delta = 0$ and α any complex number was analyzed in [10] for its relevant applications to Frobenius manifolds and quantum cohomology.

To summarize the results for the generic case, we first observe that the critical behaviors provided by the elliptic representations along regular paths (except special directions for $\mathcal{V} = 0, 1$, see Theorem 2) at $x = 0$, $x = 1$ and $x = \infty$ respectively (see [11] for $x = 1, \infty$) are

$$y(x) = a^{(0)} x^{\nu_2^{(0)}} (1 + \text{ higher orders in } x), \quad x \to 0 \tag{13}$$

$$y(x) = 1 - a^{(1)} (1-x)^{\nu_2^{(1)}} (1 + \text{ higher orders in } (1-x)), \quad x \to 1 \tag{14}$$

$$y(x) = a^{(\infty)} x^{1-\nu_2^{(\infty)}} (1 + \text{ higher orders in } x^{-1}), \quad x \to \infty \tag{15}$$

and the parameters $\nu_1^{(i)}$ are given by

$$e^{i\pi\nu_1^{(0)}} = -4a^{(0)}\, 16^{\nu_2^{(0)}-1}, \quad e^{-i\pi\nu_1^{(1)}} = -4a^{(1)}\, 16^{\nu_2^{(1)}-1}, \quad e^{i\pi\nu_1^{(\infty)}} = -4a^{(\infty)}\, 16^{\nu_2^{(\infty)}-1}$$

If $\nu_2^{(i)}$ is real, the behavior is as above when $0 < \nu_2^{(i)} < 1$. Otherwise, when $1 < \nu_2^{(i)} < 2$ it is:

$$y(x) = a^{(0)} x^{2-\nu_2^{(0)}} (1 + \text{ higher orders in } x), \quad x \to 0 \tag{16}$$

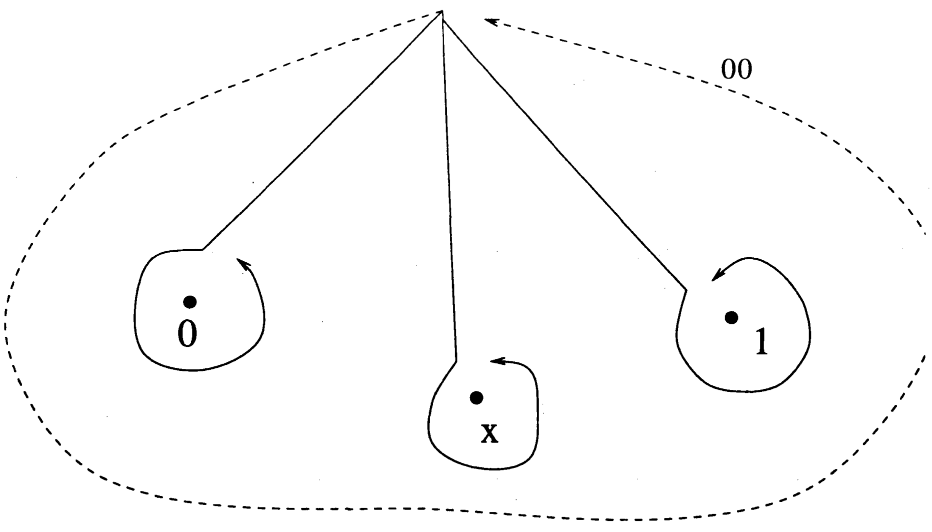

Figure 2: The order of the basis of loops of the Fuchsian system.

$$y(x) = 1 - a^{(1)}(1-x)^{2-\nu_2^{(1)}}(1+ \text{ higher orders in } (1-x)), \quad x \to 1 \tag{17}$$

$$y(x) = a^{(\infty)} x^{\nu_2^{(\infty)}-1}(1+ \text{ higher orders in } x^{-1}), \quad x \to \infty \tag{18}$$

with

$$e^{-i\pi\nu_1^{(0)}} = -4a^{(0)}\, 16^{1-\nu_2^{(0)}}, \quad e^{i\pi\nu_1^{(1)}} = -4a^{(1)}\, 16^{1-\nu_2^{(1)}}, \quad e^{-i\pi\nu_1^{(\infty)}} = -4a^{(\infty)}\, 16^{1-\nu_2^{(\infty)}} \tag{19}$$

Note that the ambiguity $\nu_1^{(i)} \mapsto \nu_1^{(i)} + 2k$, k integer, is natural, because $v^{(i)}(x)$ does not change and the $\wp$-function is periodic.

Let M_0, M_1, M_x be the monodromy matrices at $z = 0, 1, x$, for a given basis in the fundamental group of $\mathbf{P}^1 \backslash \{0, 1, x, \infty\}$. Such basis is chosen as in figure 2.

If

$$\theta_0,\ \theta_x,\ \theta_1,\ \theta_\infty\ \notin \mathbf{Z}$$

there is a one to one correspondence between a given choice of *monodromy data* θ_0, θ_x, θ_1, θ_∞, $\mathrm{tr}(M_0 M_x)$, $\mathrm{tr}(M_0 M_1)$, $\mathrm{tr}(M_1 M_x)$ and a transcendent $y(x)$ (see[13] [6], [10]). Namely:

$$y(x) = y\big(x;\ \theta_0, \theta_x, \theta_1, \theta_\infty, \mathrm{tr}(M_0 M_x), \mathrm{tr}(M_0 M_1), \mathrm{tr}(M_1 M_x)\ \big) \tag{20}$$

We proved that such a transcendent has elliptic representations at $x = 0, 1, \infty$, provided that (12) is satisfied. The three sets of parameters $(\nu_1^{(i)}, \nu_2^{(i)})$, $i = 0, 1, \infty$ are functions of the monodromy data θ_0, θ_x, θ_1, θ_∞, $\mathrm{tr}(M_0 M_x)$, $\mathrm{tr}(M_0 M_1)$, $\mathrm{tr}(M_1 M_x)$. Namely, we showed that

$$2\cos(\pi\nu_2^{(0)}) = -\mathrm{tr}(M_0 M_x), \quad 2\cos(\pi\nu_2^{(1)}) = -\mathrm{tr}(M_1 M_x), \quad 2\cos(\pi\nu_2^{(\infty)}) = -\mathrm{tr}(M_0 M_1) \tag{21}$$

$$a^{(i)} = a^{(i)}\big(\nu_2^{(i)}; \theta_0, \theta_x, \theta_1, \theta_\infty, \mathrm{tr}(M_0 M_x), \mathrm{tr}(M_0 M_1), \mathrm{tr}(M_1 M_x)\big), \quad i = 0, 1, \infty \tag{22}$$

The formulas of $a^{(i)}$ are quite long, so we do not write them here. They depend on the monodromy data through rational, trigonometric and Γ-functions. In particular, $\nu_2^{(i)}$ enters explicitly. The procedure for computing such formulae is given in the Appendix of [11]. We note that the condition $\nu_2^{(i)} \notin \mathbf{Z}$ is equivalent to $\mathrm{tr}(M_i M_j) \neq \pm 2$.

Conversely, we proved that a transcendent $y(x)$ given by its elliptic representation, under the conditions of Theorem 1 (and Theorem 3 of [11]), is a transcendent (20). This follows from the consideration that the couple $(\nu_1^{(i)}, \nu_2^{(i)})$ is given at the critical point $x = i$, and θ_0, θ_x, θ_1, θ_∞ are fixed by the equation PVI we are considering. From these data we can compute $\mathrm{tr}(M_0 M_x)$, $\mathrm{tr}(M_1 M_x)$, $\mathrm{tr}(M_0 M_1)$. One of the traces is $-2\cos(\pi\nu_2^{(i)})$, the others depend on $\nu_1^{(i)}, \nu_2^{(i)}$, θ_0, θ_x, θ_1, θ_∞ through rational, trigonometric and Γ-functions. The formulae are rather long, so we refer the reader to the Appendix of [11]. In this way the transcendent (20) is obtained. From the monodromy data we compute the couples $(\nu_1^{(j)}, \nu_2^{(j)})$

at the other two critical points and we get the the elliptic representation of the initial transcendent at the other critical points. Therefore, the connection problem is solved.

Note that if we start from the elliptic representation at one critical point, say for example $x = 0$, then $\nu_1^{(0)}, \nu_2^{(0)}$ are given. As explained above, we can compute the monodromy data and from them we compute $\nu_2^{(j)}$ and $a^{(j)}$ (then $\nu_1^{(j)}$) at the other two critical points. As already observed, the ambiguity $\nu_1^{(j)} \mapsto \nu_1^{(j)} + 2k$ (k integer) does not change the elliptic representation. On the other hand, the ambiguities $\nu_2^{(j)} \mapsto \nu_2^{(j)} + 2N$ (N integer), $\nu_2^{(j)} \mapsto -\nu_2^{(j)}$ and the ambiguity in the choice $0 \leq \Re\nu_2^{(j)} \leq 1$ or $1 \leq \Re\nu_2^{(j)} \leq 2$, which results from the cosines in (21), is due to the fact that the same transcendent has different elliptic representations in different domains (the choice of $\nu_2^{(j)}$ determines the representation and the domain!).

To summarize the results, we say that :

In the generic case (12) there is a one-to-one correspondence between monodromy data and transcendents (20). If tr$(M_iM_j) \neq \pm 2$ they have elliptic representation whose parameters $(\nu_1^{(i)}, \nu_2^{(i)})$ are given by the formulae (21), (22), (19). Conversely, a transcendent whose elliptic representation satisfies the conditions of Theorem 1 (and Theorem 3 of [11]) is a transcendet (20). The connection between its three pairs $(\nu_1^{(i)}, \nu_2^{(i)})$ is explained above. This solves the connection problem.

To conclude the discussion of the generic case, some comments about our extension of previous known results are in order. The critical behavior for a class of solutions to the Painlevé 6 equation was found by Jimbo in [13] for generic values of α, β, γ δ. A transcendent in this class has behavior:

$$y(x) = a^{(0)} x^{1-\sigma^{(0)}}(1 + O(|x|^\delta)), \quad x \to 0, \tag{23}$$

$$y(x) = 1 - a^{(1)}(1-x)^{1-\sigma^{(1)}}(1 + O(|1-x|^\delta)), \quad x \to 1, \tag{24}$$

$$y(x) = a^{(\infty)} x^{-\sigma^{(\infty)}}(1 + O(|x|^{-\delta})), \quad x \to \infty, \tag{25}$$

where δ is a small positive number, $a^{(i)}$ and $\sigma^{(i)}$ are complex numbers such that $a^{(i)} \neq 0$ and

$$0 \leq \Re\sigma^{(i)} < 1. \tag{26}$$

We remark that x converges to the critical points *inside a sector* with vertex on the corresponding critical point. The connection problem, i.e. the problem of finding the relation among the three pairs $(\sigma^{(i)}, a^{(i)})$, $i = 0, 1, \infty$, was solved in [13] for the above class of transcendents using the isomonodromy deformations theory. Actually, a transcendent in the class above coincides with a transcendent (20). In particular

$$2\cos(\pi\sigma^{(0)}) = \mathrm{tr}(M_0M_x), \quad 2\cos(\pi\sigma^{(1)}) = \mathrm{tr}(M_1M_x), \quad 2\cos(\pi\sigma^{(\infty)}) = \mathrm{tr}(M_0M_1) \tag{27}$$

and

$$a^{(i)} = a^{(i)}\big(\sigma^{(i)}; \theta_0, \theta_x, \theta_1, \theta_\infty, \mathrm{tr}(M_0M_x), \mathrm{tr}(M_0M_1), \mathrm{tr}(M_1M_x)\big), \quad i = 0, 1, \infty$$

For the formulas of $a^{(i)}$ we refer to [13]. The monodromy data are restricted by the following condition, equivalent to (26):

$$|\mathrm{tr}(M_iM_j)| \leq 2, \quad \Re\{\mathrm{tr}(M_iM_j)\} \neq -2 \tag{28}$$

As explained above, we have shown that the transcendents (20) have elliptic representation. Therefore, Jimbo's transcendents are included in our class of transcendents obtained by the elliptic representation. Observe that the behaviors (23)–(25) are included in the behaviors (13)–(15) with $\sigma^{(i)} = 1 - \nu_2^{(i)}$ (and (16)–(18) with $\sigma^{(i)} = \nu_2^{(i)} - 1$). We proved in [11] that the condition (26) is extended to any $\sigma^{(i)} \in \mathbf{C}$ such that $\sigma^{(i)} \notin (-\infty, 0] \cup [1, +\infty)$ (as we must expect, if we observe that $\nu_2^{(i)} \notin (-\infty, 0] \cup \{1\} \cup [2, +\infty)$ and that (27) defines $\sigma^{(i)}$ up to $\sigma^{(i)} \mapsto \pm\sigma^{(i)} + 2n$, n integer). Therefore we have solved the connection problem for any complex value of $\mathrm{tr}(M_iM_j)$ with the only constraint $\mathrm{tr}(M_iM_j) \neq \pm 2$. This condition extends (28).

To be more precise, the condition $\nu_2^{(i)} \neq 1$ is equivalent to $\mathrm{tr}(M_0M_x) \neq 2$ at $x = 0$; to $\mathrm{tr}(M_1M_x) \neq 2$ at $x = 1$; to $\mathrm{tr}(M_0M_1) \neq 2$ at $x = \infty$. Nevertheless, in the case $\mathrm{tr}(M_iM_j) = 2$ the critical behavior and the solution of the connection problem were achieved by Jimbo. Unfortunately, the condition $\nu_2^{(i)} \neq 1$ which we had to impose to study the elliptic representation (except for non-generic cases like

$\beta = \gamma = 1 - 2\delta = 0$) does not allow us to know the analytic properties and the critical behavior of the elliptic representation in this case. We expect that the properties of $u(x)$ are such to exactly produce the critical behavior found by Jimbo for $\mathrm{tr}(M_i M_j) = 2$, but we still have to cover this case.

The condition $\nu_2^{(i)} \neq 0$ (and 2), implies that we can not give the critical behaviors (and the elliptic representation) of (20) at $x = 0$ for $\mathrm{tr}(M_0 M_x) = -2$; at $x = 1$ for $\mathrm{tr}(M_1 M_x) = -2$; at $x = \infty$ for $\mathrm{tr}(M_0 M_1) = -2$. To our knowledge, these cases have not yet been studied in the literature.

To conclude, the results of [13] together with our extension provide the critical behaviors and the solution of the connection problem for the transcendents (20) in the generic case for

$$\text{any value of } \mathrm{tr}(M_i M_j) \neq -2$$

which corresponds to exponents

$$\sigma^{(i)} \in \mathbf{C} \text{ such that } \sigma^{(i)} \notin (-\infty, 0) \cup [1, +\infty).$$

We turn now to the special case $\beta = \gamma = 1 - 2\delta = 0$, important for its applications to topological filed theory, Frobenius manifolds [4] and quantum cohomology [17] [12]. This case is fully studied in [10]. We can give a representation of $u(x)$ in a domain which is wider than the generic case. Namely, at $x = 0$, the domain is

$$\mathcal{D}(r; \nu_1, \nu_2) := \left\{ x \in \tilde{\mathbf{C}}_0 \mid |x| < r, \ \left| e^{-i\pi\nu_1} \left(\frac{x}{16}\right)^{2-\nu_2} \right| < r, \ \left| e^{i\pi\nu_1} \left(\frac{x}{16}\right)^{\nu_2} \right| < r \right\}$$

In this domain $v(x)$ is holomorphic with convergent expansion

$$v(x) = \sum_{n\geq 1} a_n x^n + \sum_{n\geq 0, m\geq 1} b_{nm} x^n \left[e^{-i\pi\nu_1} \left(\frac{x}{16}\right)^{2-\nu_2} \right]^m + \sum_{n\geq 0, m\geq 1} c_{nm} x^n \left[e^{i\pi\nu_1} \left(\frac{x}{16}\right)^{\nu_2} \right]^m$$

If ν_2 is real, the value $\nu_2 = 1$ is now allowed, namely, the constraint is $\nu_2 \notin (-\infty, 0] \cup [2, +\infty)$. Therefore, by periodicity of the $\wp$-function we can assume $0 \leq \Re\nu_2 < 2$, $\nu_2 \neq 0$. A similar result holds at $x = 1$ and $x = \infty$.

According to [6], we define $2 - x_0^2 := \mathrm{tr}\ M_0 M_x$, $2 - x_1^2 := \mathrm{tr}\ M_1 M_x$, $2 - x_\infty^2 := \mathrm{tr}\ M_0 M_1$. There is a one to one correspondence between triples (x_0, x_1, x_∞) (defined up to the change of two signs) and Painlevé transcendents, provided that at most one x_i is zero and not all the x_i are ± 2 at the same time. Therefore we write $y(x) = y(x; x_0, x_1, x_\infty)$. We show that one such transcendent has elliptic representations (half-periods are understood)

$$y(x; x_0, x_1, x_\infty) = \wp(\nu_1^{(0)} \omega_1^{(0)}(x) + \nu_2^{(0)} \omega_2^{(0)}(x) + v^{(0)}(x; \nu_1^{(0)}, \nu_2^{(0)})) + \frac{1+x}{3}$$

$$= \wp(\nu_1^{(1)} \omega_1^{(1)}(x) + \nu_2^{(1)} \omega_2^{(1)}(x) + v^{(1)}(x; \nu_1^{(1)}, \nu_2^{(1)})) + \frac{1+x}{3} \tag{29}$$

$$= \wp(\nu_1^{(\infty)} \omega_1^{(\infty)}(x) + \nu_2^{(\infty)} \omega_2^{(\infty)}(x) + v^{(\infty)}(x; \nu_1^{(\infty)}, \nu_2^{(\infty)})) + \frac{1+x}{3} \tag{30}$$

The parameters $\nu_2^{(i)}$ are obtained from

$$\cos \pi\nu_2^{(i)} = \frac{x_i^2}{2} - 1, \quad 0 \leq \Re\nu_2^{(i)} \leq 1, \quad \nu_2^{(i)} \neq 0, \qquad i = 0, 1, \infty$$

Note that the condition $x_i \neq \pm 2$, $i = 0, 1, \infty$, corresponds to $\nu_2^{(i)} \neq 0$. The parameter $\nu_1^{(0)}$ is obtained by the formula

$$e^{i\pi\nu_1^{(0)}} = -\frac{i\Gamma^4\left(1 - \frac{\nu_2^{(0)}}{2}\right)}{2\sin(\pi\nu_2^{(0)})\Gamma^2\left(\frac{3}{2} - \mu - \frac{\nu_2^{(0)}}{2}\right)\Gamma^2\left(\frac{1}{2} + \mu - \frac{\nu_2^{(0)}}{2}\right)} \Big[2(1 - e^{i\pi\nu_2^{(0)}}) -$$

$$- f(x_0, x_1, x_\infty)(x_\infty^2 - e^{i\pi\nu_2^{(0)}} x_1^2)\Big] f(x_0, x_1, x_\infty)$$

where

$$f(x_0,x_1,x_\infty) := \frac{4-x_0^2}{x_1^2+x_\infty^2-x_0x_1x_\infty}, \quad \alpha = \frac{(2\mu-1)^2}{2}$$

Moreover, $\exp\{-i\pi\nu_1^{(1)}\}$, $\exp\{i\pi\nu_1^{(\infty)}\}$ are given by an analogous formula with the substitutions $(x_0,x_1,x_\infty) \mapsto (x_1,x_0,x_0x_1-x_\infty)$, $\nu_2^{(0)} \mapsto \nu_2^{(1)}$ and $(x_0,x_1,x_\infty) \mapsto (x_\infty,-x_1,x_0-x_1x_\infty)$, $\nu_2^{(0)} \mapsto \nu_2^{(\infty)}$ respectively.

The most general choice of ν_2 is $0 \leq \Re\nu_2 < 2$. This corresponds to the fact that the transcendent $y(x;x_0,x_1,x_\infty)$ also has three representations

$$\begin{aligned} y(x;x_0,x_1,x_\infty) &= \wp(\tilde{\nu}_1^{(0)}\omega_1^{(0)}(x)+\tilde{\nu}_2^{(0)}\omega_2^{(0)}(x)+v^{(0)}(x;\tilde{\nu}_1^{(0)},\tilde{\nu}_2^{(0)}))+\frac{1+x}{3} \\ &= \wp(\tilde{\nu}_1^{(1)}\omega_1^{(1)}(x)+\tilde{\nu}_2^{(1)}\omega_2^{(1)}(x)+v^{(1)}(x;\tilde{\nu}_1^{(1)},\tilde{\nu}_2^{(1)}))+\frac{1+x}{3} \\ &= \wp(\tilde{\nu}_1^{(\infty)}\omega_1^{(\infty)}(x)+\tilde{\nu}_2^{(\infty)}\omega_2^{(\infty)}(x)+v^{(\infty)}(x;\tilde{\nu}_1^{(\infty)},\tilde{\nu}_2^{(\infty)}))+\frac{1+x}{3} \end{aligned}$$

where

$$\cos\pi\tilde{\nu}_2^{(i)} = \frac{x_i^2}{2}-1, \quad 1 \leq \Re\nu_2^{(i)} < 2, \quad i=0,1,\infty$$

The parameter $\tilde{\nu}_1^{(0)}$ is obtained by the formula

$$e^{-i\pi\tilde{\nu}_1^{(0)}} = \frac{i\Gamma^4\left(\frac{\tilde{\nu}_2^{(0)}}{2}\right)}{2\sin(\pi\tilde{\nu}_2^{(0)})\Gamma^2\left(\frac{1}{2}-\mu+\frac{\tilde{\nu}_2^{(0)}}{2}\right)\Gamma^2\left(-\frac{1}{2}+\mu+\frac{\nu_2}{2}\right)}\Big[2(1-e^{-i\pi\tilde{\nu}_2^{(0)}})-$$

$$-f(x_0,x_1,x_\infty)(x_\infty^2-e^{-i\pi\tilde{\nu}_2^{(0)}}x_1^2)\Big]\,f(x_0,x_1,x_\infty).$$

$\exp\{i\pi\tilde{\nu}_1^{(1)}\}$, $\exp\{-i\pi\tilde{\nu}_1^{(\infty)}\}$ are given by an analogous formula with the substitutions $(x_0,x_1,x_\infty) \mapsto (x_1,x_0,x_0x_1-x_\infty)$, $\tilde{\nu}_2^{(0)} \mapsto \tilde{\nu}_2^{(1)}$ and $(x_0,x_1,x_\infty) \mapsto (x_\infty,-x_1,x_0-x_1x_\infty)$, $\tilde{\nu}_2^{(0)} \mapsto \tilde{\nu}_2^{(\infty)}$ respectively. The formulae above have limits for $\nu_2 = 1, 1 \pm 2\mu + 2m$, m integer. They are listed in [10] and [11].

Conversely, a transcendent

$$y(x) = \wp(\nu_1\omega_1^{(0)}(x)+\nu_2\omega_2^{(0)}(x)+v^{(0)}(x;\nu_1,\nu_2))+\frac{1+x}{3}, \quad \text{at } x=0 \tag{31}$$

coincides with $y(x;\ x_0,x_1,x_\infty)$, with the following monodromy data.

If $0 \leq \Re\nu_2 \leq 1$:

$$\begin{aligned} x_0 &= 2\cos\left(\frac{\pi}{2}\nu_2\right) \\ x_1 &= \left[\frac{4^{-\nu_2}\,2\,e^{i\frac{\pi}{2}\nu_1}}{f(\nu_2,\mu)G(\nu_2,\mu)}+\frac{G(\nu_2,\mu)}{4^{-\nu_2}\,2\,e^{i\frac{\pi}{2}\nu_1}}\right] \\ x_\infty &= \left[\frac{4^{-\nu_2}\,2\,e^{i\frac{\pi}{2}(\nu_1-\nu_2)}}{f(\nu_2,\mu)G(\nu_2,\mu)}+\frac{G(\nu_2,\mu)}{4^{-\nu_2}\,2\,e^{i\frac{\pi}{2}(\nu_1-\nu_2)}}\right] \end{aligned}$$

where

$$f(\nu_2,\mu) = -\frac{2\sin^2\left(\frac{\pi}{2}\nu_2\right)}{\cos(\pi\nu_2)+\cos(2\pi\mu)}, \quad G(\nu_2,\mu) = 4^{-\nu_2}\,2\,\frac{\Gamma\left(1-\frac{\nu_2}{2}\right)^2}{\Gamma\left(\frac{3}{2}-\mu-\frac{\nu_2}{2}\right)\Gamma\left(\frac{1}{2}+\mu-\frac{\nu_2}{2}\right)}$$

If $1 \leq \Re\nu_2 < 2$:

$$\begin{aligned} x_0 &= 2\cos\left(\frac{\pi}{2}\nu_2\right) \\ x_1 &= \left[\frac{e^{-i\frac{\pi}{2}\nu_1}}{4^{1-\nu_2}\,2\,f(\nu_2,\mu)G_1(\nu_2,\mu)}+\frac{4^{1-\nu_2}\,2\,G_1(\nu_2,\mu)}{e^{-i\frac{\pi}{2}\nu_1}}\right] \end{aligned}$$

$$x_\infty = \left[\frac{e^{i\frac{\pi}{2}(\nu_2-\nu_1)}}{4^{1-\nu_2}\ 2\ f(\nu_2,\mu)G_1(\nu_2,\mu)} + \frac{4^{1-\nu_2}\ 2\ G_1(\nu_2,\mu)}{e^{i\frac{\pi}{2}(\nu_2-\nu_1)}}\right]$$

where

$$G_1(\nu_2,\mu) = \frac{1}{4^{1-\nu_2}\ 2}\frac{\Gamma\left(\frac{\nu_2}{2}\right)^2}{\Gamma\left(\frac{1}{2}-\mu+\frac{\nu_2}{2}\right)\Gamma\left(-\frac{1}{2}+\mu+\frac{\nu_2}{2}\right)}$$

After computing the monodromy data, we can write the elliptic representations of $y(x;x_0,x_1,x_\infty)$ at $x=1$ and $x=\infty$, namely (29), (30). Since they are the elliptic representations at $x=1$, $x=\infty$ of (31), we have solved the connection problem for (31).

We observed that there is a one to one correspondence between Painlevé transcendents and triples of monodromy data (x_0,x_1,x_∞), defined up to the change of two signs, satisfying $x_i \neq \pm 2$, $i=0,1,\infty$, i.e. $\nu_2^{(i)} \neq 0$ (and 2), and at most one $x_i = 0$. The cases when these conditions are not satisfied are studied in [19]. However, if $x_i = \pm 2$ (namely the trace is -2) the problem of finding the critical behavior at the corresponding critical point $x=i$ is still open (except when *all the three* x_i are ± 2: in this case there is a one-parameter class of solutions called *Chazy solutions* in [19]). We conclude that the results of our papers [10] [11] plus the results of [19] cover all the possible transcendents, except the special case when one or two x_i are ± 2.

Finally, we expect that in all non-generic cases we can solve the connection problem and express the parameters ν_1, ν_2 in terms of monodromy data. From the conceptual point of view nothing should change with respect to [13] [6] [10] [11]; but the technical details may require a long time for computations.

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